Applications of First Order Differential Equations

Your report on this assignment should conform to the requirements given in the information sheet; in particular the questions (and parts thereof) are to be answered in increasing order with MAPLE calculations alternating with relevant commentary (no "MAPLE appendices" permitted). You should describe fully how you solved these modeling problems. Use the text insertion capabilities of MAPLE where appropriate; in particular you should add text briefly describing each MAPLE command before you execute it. The MAPLE file `ass3-fall20.mw` should be downloaded from Canvas and used as a command template.

1. **A Chemical Reaction.**

   Two chemicals X and Y react in such a way that one gram of X combines with 4 grams of Y to produce a compound Z which is written by the chemists as

   $$X + 4Y \to Z.$$

   The law of mass action in chemical kinetics states that, at a given time t, the rate at which X and Y react (i.e. the rate at which Z is produced) is proportional to the product of the amounts of X and Y that remain untransformed at that time (so this is a second order reaction). Assume that there are initially 50 grams of X and 32 grams of Y, and it is known that 30 grams of Z are formed in 10 minutes.

   (a) Let $z(t)$ grams of Z be present at time t. Derive a differential equation initial value problem for $z(t)$ and solve it by hand calculation.

   (b) Check your solution in (a) using MAPLE.

   (c) How much of the compound Z is present after 15 minutes?

   (d) How much Z is present after a very long time ($t \to \infty$)?

2. **The Corporate Boardroom.**

   A corporate boardroom contains 4500 ft^3 of air initially free of carbon monoxide. Each morning (time $t = 0$) the executives arrive and cigarette smoke containing 4% carbon monoxide by volume is introduced into the room at the rate of 0.3 ft^3/min by the smokers. A ceiling fan keeps the air in the room well circulated, and the air and smoke mixture leaves the room at the rate of 10 ft^3/min. After exiting the room the carbon monoxide is removed from the air and smoke mixture via a catalytic converter system before being returned as clean air to the room at the same rate by the building's forced-air system.

Figure 1: The corporate board room

(a) Explain in detail how to find a linear differential equation for the number of cubic feet, $A(t)$, of carbon monoxide in the room at time t minutes. Using the initial condition $A(0) = 0$, solve this equation by hand to determine a formula for $A(t)$.

(b) Check your result in (a) by solving the initial value problem using MAPLE.

(c) Use the MAPLE `plot()` command to plot the carbon monoxide *concentration* over the t range $[0, 10000]$. Observe that, to activate the solution $A(t)$ as a MAPLE function (and avoid `''empty plot''` errors), you will need to use the MAPLE `unapply()` and `rhs()` functions as we did in class. From your solution $A(t)$ use the MAPLE `fsolve()` command to find how long before the concentration of carbon monoxide becomes 0.02%.

(d) It is known that a carbon monoxide concentration of 0.16% is lethal. Can such life-threatening concentrations appear in the conference room? It is also known that exposure to concentrations at or above 0.02% are associated with loss of judgement on the part of the subjects. Since the occupants of the room are the senior executives in the company, do you have any recommendations for savvy investors holding stock in the company?

3. **Linear First Order Differential Equation**

The *general solution* of a first-order differential equation is a formula, containing one arbitrary constant, representing all solutions of the equation. Find, by hand, showing all your working, the general solution of the first-order linear equation

$$(x^2 - 1)\frac{dy}{dx} - xy = 2x(x^2 - 1).$$

Hint: be careful, this equation is not yet in "standard form".