Inversion Theory
Assignment 2

1. Linear dependence

Prove that any 4 vectors ($\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$,$\mathbf{d}$) in three-dimensional Euclidean space are linearly dependent.

2. Normed linear spaces

A normed linear space can be made a metric space if we introduce a metric by the formula:

$$\mu\,(\mathbf{f},\mathbf{g}) = \|\mathbf{f}-\ \mathbf{g}\|$$

where $\mu\,(\mathbf{f},\mathbf{g})$ is the distance between $\mathbf{f}$ and $\mathbf{g}$.

Prove: Expression $\mu\,(\mathbf{f},\mathbf{g})$ has all properties of the metric, according to the definition of the metric space.

3. Basis of the Hilbert space L_2.

Prove that the system of functions

$$\left[\frac{1}{\sqrt{2\pi}},\frac{\cos\mathbf{nx}}{\sqrt{\pi}},\frac{\sin\mathbf{nx}}{\sqrt{\pi}}\right],\ n=1,2,.....$$

forms the orthonormal set of elements in the Hilbert space $L_2(-\pi,\pi)$.

4. Minimization problem in Hilbert space

Problem:

Determine the function $d = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + ... + \alpha_7 x^6$, which fits the best to the observed data d_0 in the interval (0, 3.5). In another words, solve the nonlinear regression problem. Data set $d_0 = f(x)$ is given in the table:

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	2	1	1.5	2	3	4	5	5.4	5.7	5.3	5	4.4	3.9	3	2.5	2.2	2	2.7	3

x	1.9	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3	3.1	3.2	3.3	3.4	3.5
$f(x)$	4	5	6	7	8	8.5	8.7	9	7.5	6	4.5	2.9	1	−0.1	−0.3	1.9	5

Plot data values and fitted curve on the same plot. Include program listing. Also, print polynomial coefficients α_i and the corresponding Gram matrix.

Explanation:

Let us assume that L is subspace of a Hilbert space $L_2\ (\ 0, 3.5)$ spanned over a linearly independent set of seven vectors $\{1, x, x^2, x^3, x^4, x^5, x^6\}$. The problem is for experimental set of data d_0 (described above in the table) to determine the vector $d \in L$, closest to d_0.

For solving this problem we consider a norm of difference $\|d_0 - d\|$. Any vector $d \in L$ can be represented in the form of a linear combination of the linearly independent vectors:

$$d = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + ... + \alpha_7 x^6. \tag{1}$$

Thus, we arrive at the minimization problem,

$$\|d_0 - d\|^2 = \|d_0 - (\alpha_1 + \alpha_2 x + \alpha_3 x^2 + ... + \alpha_7 x^6)\|^2 = \min, \tag{2}$$

or, using the inner product notation:

$$\|d_0 - d\|^2 = \|d_0 - \sum_{i=1}^{7} \alpha_i x^{i-1}\|^2 =$$

$$= (d_0 - \sum_{i=1}^{7} \alpha_i x^{i-1}, d_0 - \sum_{i=1}^{7} \alpha_i x^{i-1}) = \min .$$

Let us calculate the derivatives of the $\|d_0 - d\|^2$ with respect to α_j . These derivatives must vanish at extremum point:

$$\frac{\partial}{\partial \alpha_j} \|d_0 - d\|^2 = 2(d_0 - \sum_{i=1}^{7} \alpha_i x^{i-1}, x^{j-1}) = 0.$$

From the last equation we obtain a system of the linear equations for the unknown coefficients α_i:

$$\sum_{i=1}^{7} \alpha_i (x^{i-1}, x^{j-1}) = (d_0, x^{j-1}). \tag{3}$$

We may write the system of equations in a more compact form as:

$$\sum_{i=1}^{7} \Gamma_{ji}\alpha_i = (d_0, x^{j-1}),$$

where the symmetric matrix Γ_{ji} is the corresponding *Gram* matrix:

$$\Gamma_{ji} = (x^{i-1}, x^{j-1}). \tag{4}$$

Directions:

Use Matlab. To write the code, implementing this assignment, it is convenient to reformulate the problem using matrix notations.

Note that, the inner product of two functions, $p(x)$ and $q(x)$, in a case of the discrete observations in 36 points $(x_1,\ x_2,\ .....x_{36})$, will reduce to the dot product of two vectors $\mathbf{p} = (p(x_1),\ p(x_2),\ .....p(x_{36}))$ and $\mathbf{q} = (q(x_1),\ q(x_2),\ .....q(x_{36}))$:

$$(p(x), q(x)) = \mathbf{p} \cdot \mathbf{q} = \sum_{i=1}^{36} p(x_i)\, q(x_i)\,.$$

Expression (1) can be written as:

$$\mathbf{Fm} = \mathbf{d} \tag{5}$$

where $\mathbf{m}$ is the vector-column containing coefficients α, (or vector of model parameters), $\mathbf{F}$ is the Vandermonde matrix (which contains the powers of x), and $\mathbf{d}$ is a vector-column of data:

$$F = \begin{bmatrix} 1 & x_1 & x_1^2 & ... & x_1^6 \\ ... & ... & ... & ... & ... \\ 1 & x_i & x_i^2 & ... & x_i^6 \\ ... & ... & ... & ... & ... \\ 1 & x_{36} & x_{36}^2 & ... & x_{36}^6 \end{bmatrix}, \quad m = \begin{bmatrix} \alpha_1 \\ ... \\ \alpha_i \\ ... \\ \alpha_7 \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ ... \\ d_i \\ ... \\ d_{36} \end{bmatrix}. \tag{6}$$

In this notations, Gram matrix $\mathbf{G}$ (consisting of elements Γ_{ji}, given by formula (4)) becomes a square (7×7) matrix:

$$\mathbf{G} = \mathbf{F}^T\mathbf{F},$$

and, equation (3) becomes:

$$\mathbf{Gm} = \mathbf{F}^T\mathbf{d}_0, \tag{7}$$

where

$$\mathbf{d}_0 = \begin{bmatrix} f(x_1) \\ ... \\ f(x_i) \\ ... \\ f(x_{36}) \end{bmatrix}.$$

Letter "T" in the upper subscribe denotes transposition of a matrix or a vector. Note that the system (7) has seven equations and seven unknowns. Thus, it can be directly inverted.

Write the part of the code, which fills out the Vandermonde matrix. Hint: first create vector-column $\mathbf{x}$, than concatenate seven vectors containing powers of x to each other. Matlab vectorized arithmetic has function "dot-hat" denoted as ".^", which allows to raise each element of vector into a given power. To concatenate columns, use [ ] operator, for example: $[x.\hat{}1\ x.\hat{}2]$.

Fill out the vector of data $\mathbf{d}_0$ and solve equation (7) using Matlab function "inv".

Create required printouts.