MATH 645 Exam I

Available until October 1 2020

This exam is 7 pages long, including this cover sheet, and consists of 9 mandatory questions and one optional bonus question. Before you begin, verify that you have the correct number of pages and questions, then put your name on the line below and on the upper-right hand corner of each of following pages. Answer each question in the spaces provided. If you run out of room for an answer, continue on the back of one of the pages and clearly indicate under the question where your work can be found. You must show all your work in order to receive full credit for a solution. You may use a four-function or scientific calculator. Read the instructions for each question thoroughly, and double-check your solutions before you hand in your exam. Good luck!

Name: ______________________________

Instructor: Jeremiah Johnson

Points Per Problem	Score
1: 6	
2: 6	
3: 2	
4: 4	
5: 2	
6: 8	
7: 3	
8: 3	
9: 3	
10: 3 (bonus, optional)	
Total: 37 Possible	

1. Use the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ for the questions below.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

(a) (2 points) Calculate the dot products $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{w}$ or explain why they do not exist.

(b) (3 points) Calculate the norms $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ of $\mathbf{u}$ and $\mathbf{v}$.

(c) (2 points) Using your answers to parts (a) and (b) above, confirm that Schwarz's Inequality holds for $\mathbf{u}$ and $\mathbf{v}$ (in other words, verify that $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$).

2. Use the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ for the questions below.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

(a) (3 points) Are $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ linearly independent or linearly dependent? Justify/explain your response.

(b) (3 points) Consider the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$. In $\mathbb{R}^3$, this set of vectors defines what type of geometric object?

3. (2 points) Write the following linear combination as a matrix-vector multiplication:

$$c_1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \mathbf{b}$$

4. (4 points) Solve the following system of equations using elimination.

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

5. (2 points) What is the connection between question 3 and question 4?

6. Using the matrices A, B, and C given below, perform the following calculations or explain why they cannot be done.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) (2 points) $A + C$

(b) (3 points) $A \cdot B$

(c) (3 points) $B^T B$

7. (3 points) For the matrix A given below, produce the elimination matrices E_{21}, E_{31}, and E_{32} to make A upper triangular; that is, produce E_{21}, E_{31}, and E_{32} such that $E_{32}E_{31}E_{21}A = U$.

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 0 & 7 \\ -1 & 7 & 10 \end{bmatrix}$$

8. (3 points) Produce the LU factorization of the matrix A below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

9. (3 points) Refer to A, B, and C in question 6. Both A, C, and $B^T B$ are what special type of matrix? (Hint: what are their transposes?) For this type of matrix, what is special about their LDU factorization? Illustrate by calculating the LDU factorization of A.

10. (Bonus, 3 points) A band matrix is a square matrix with nonzero entries only on the main diagonal and on w of the diagonals above and below the main diagonal, and zeros everywhere else. The matrix B below is an example of a 5x5 symmetric band matrix with $w = 1$. Elimination for band matrices is much cheaper than elimination for ordinary matrices. For a generic $n \times n$ band matrix B with exactly w nonzero bands above and below the main diagonal, approximately how many multiplication and subtraction operations are necessary for elimination $B \to U$?

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

End of Exam