Assignment 2

Instructions: Problems marked $*$ are to be submitted for grading through **Crowdmark**. The deadline is **Friday, Oct. 2, 23:00 (PDT)**. The remaining problems are included as practice problems, and you are encouraged to try as many as you can.

$*$**1. (10 points)** Let A and B be subgroups of a group G. Recall from Assignment 1 that $A \cap B$ is a subgroup of G, and hence of A.

(a) Show that
$$f\colon A/(A\cap B) \to G/B, \quad a(A\cap B) \mapsto aB$$
is a well-defined map, i.e., is independent of any choice of coset representatives.

(b) Show that f is injective.

(c) Deduce that $(A : A\cap B) \leq (G : B)$ in the case where $(G : B)$ is finite.

(d) Suppose now that $G = S_n$ for some integer $n \geq 2$. Show that if A contains an odd permutation, then exactly half the elements of A are odd (*Hint*: Use (c)).

$*$**2. (7 points)** Let G and H be groups.

(a) Show that if A is a subgroup of G, and B is a subgroup of H, then $A \times B$ is a subgroup of $G \times H$.

(b) Is is possible for $G \times H$ to have subgroups that are *not* of the kind described in part (a)? Justify your answer with a suitable example or proof.

3. What are the possible orders of the elements of $\mathbb{Z}_6 \times \mathbb{Z}_9$? How many elements achieve the largest possible order?

4. (4 points) For a fixed positive integer $m \geq 2$, consider the following subsets of D_{2m}:
$$A = \{\mathrm{id}, r^2, r^4, \cdots, r^{2m-2}, s, sr^2, sr^4, \ldots, sr^{2m-2}\}; \quad B = \{\mathrm{id}, r^m\}.$$

(a) Show that A and B are subgroups of D_{2m}.

(b) Show that $A \cong D_m$ and $B \cong \mathbb{Z}_2$.

$*$(c) Show that if m is *odd*, then D_{2m} is the internal direct product of A and B (and hence that $D_{2m} \cong D_m \times \mathbb{Z}_2$).

$*$**5. (8 points)** Show that if $n \geq 3$ is *odd*, then D_n is *not* the internal direct product of two *non-trivial* subgroups.

$*$**6. (7 points)** Let H and N be subgroups of a group G.

(a) Show that if N is normal, then HN is a subgroup of G.

(b) Show that if *both* N and H are normal, then HN is also normal.

$*$**7. (5 points)** Show that
$$V := \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$
is a normal subgroup of S_4.

∗**8**. **(9 points)** Show that S_4 has no normal subgroups of order 2, 3, 6 or 8.

9. Let H be a subgroup of a group G. Show that if $(G : H) = 2$, then H is normal.

10. Let H be a normal subgroup of a group G. We have seen in class that a normal subgroup of H need not be normal in G. Show, by contrast, that any *characteristic subgroup* of H is normal in G. Here, a subgroup N of H is *characteristic* if it is stable under all automorphisms of H, i.e., if $f(n) \in N$ for all $n \in N$ and $f \in \mathrm{Aut}(N)$.