1. (25) Prove that for any regular languages A and B,

$$L_1 = \{x^R y \mid x \in A, y \in B\}$$

   is regular, where x^R is the reversal of x. Prove also that

$$L_2 = \{x^R y x \mid x \in A, y \in B\}$$

   is not regular.

2. (25) Give a context-free grammer for generating the following language:

$$L_3 = \{a^i b^j c^k d^l \mid i + j = k + l\}.$$

3. (25) Prove that the following language is not context-free using the pumping lemma:

$$L_4 = \{a^i b^j c^k \mid i < j < k\}.$$

4. (25) The function $next(i, j)$ takes a pair of positive integers (i, j) as input and returns a pair of integers:

   $next(i, j) = $ **if** $(i = 1)$ **then return** $(j + 1, 1)$ **else return** $(i - 1, j + 1)$

   You are asked to implement this function by a standard Turing machine. On the tape, (i, j) is represented by the input string $\#a^i \# a^j$ (followed by blanks). Please define the Turing machine moves in details that will modify the input $\#a^i \# a^j$ into $\#a^{i'} \# a^{j'}$, where $next(i, j) = (i', j')$ and $i, j, i', j' \geq 1$. The tape head is pointing at the first symbol in the beginning. When the work is done, the tape head is also pointing at the first symbol of the tape.