1. (True or False) Please indicate whether the following statements are either **TRUE** or **FALSE**. You will receive 1 point for the correct **answer**, and one point for the correct **justification**.

   (a) (2 points) If $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^n$ satsify $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \mathbb{R}^n$, then $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a linearly independent set of vectors.

   TRUE or **FALSE**

   (b) (2 points) If A is a 2×4 matrix and B is a 4×3 matrix, then the columns of AB are linearly dependent.

   TRUE or **FALSE**

   (c) (2 points) If A,B,C are 2×2 matrices, with $A \neq 0$ (i.e. A is not the 2×2 matrix with all entries equal to 0) such that

   $$AB = AC$$

   then $B = C$.

   TRUE or **FALSE**

(d) (2 points) Every $m \times n$ matrix A is row equivalent to a unique matrix in row echelon form.

TRUE or **FALSE**

(e) (2 points) Let P be a plane in $\mathbb{R}^3$, with equation $P : ax_1 + bx_2 + cx_3 = 0$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by projection to P. Then T is invertible.

TRUE or **FALSE**

2. (10 points) Let

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \end{bmatrix}.$$

Find all 3×2 matrices X so that

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hint: Write this as a linear system of equations, with variables the entries of X. Now, row reduce to solve.

3. (10 points) Parameterize the set of all $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ so that the system of equations

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= 0 \\ ax_1 + bx_2 + cx_3 &= 1 \end{aligned}$$

has multiple solutions.

4. (a) (10 points) Suppose $\mathbf{a_1}, \mathbf{a}_2, \ldots \mathbf{a}_n \in \mathbb{R}^m$ are all non-zero vectors, and that they are all orthogonal, i.e. $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ if $i \neq j$. Show that the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\}$ is linearly independent. **Hint:** If $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots c_n\mathbf{a}_n = \mathbf{0}$ was a dependence relation, what happens when you dot with $\mathbf{a}_1$?

(b) (5 points) Without row reducing, show that the columns of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

span $\mathbb{R}^4$.

5. In this question, we will show that every rotation $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be written as a composition of two reflections.

   (a) (3 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of rotation by θ counter clockwise. Let S be the linear transformation of reflection about the x_1-axis. Give a geometric argument for why the vector

$$\mathbf{u} = \begin{bmatrix} \cos(\frac{-\theta}{2}) \\ \sin(\frac{-\theta}{2}) \end{bmatrix}$$

   satisfies

$$(S \circ T)(\mathbf{u}) = \mathbf{u}.$$

   (b) (4 points) Using the standard matrices for T and S, show that

$$(S \circ T)(\mathbf{u}) = \mathbf{u}$$

   algebraically.

(c) (6 points) Now, I claim that the linear transformation $S \circ T$ is actually reflection in the line L spanned by $\mathbf{u}$. Show this. **Hint:** You can either use the formula for reflection we found in class, or you can use some geometry. Your choice!

(d) (2 points) Explain how what we have done above shows that every rotation T can be written as a composition of two reflections.

6. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be real numbers, and consider the matrix

$$A = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{bmatrix}.$$

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be the columns of A. We will show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are linearly independent if and only if α, β, γ are all distinct from one another (i.e. $\alpha \neq \beta, \alpha \neq \gamma$, and $\beta \neq \gamma$).

(a) (10 points) Show this using row reduction.

Extra credit: Let's see this fact in a different, more conceptual way. The columns of A are linearly dependent if and only if there is a vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

so that

$$A\mathbf{c} = \mathbf{0}$$

.

(b) (2 extra credit points) Write out the actual system linear equations that encoded by the matrix equation $A\mathbf{c} = \mathbf{0}$.

(c) (2 extra credit points) Suppose $\mathbf{c}$ is such that $A\mathbf{c} = \mathbf{0}$. Consider the degree 2 polynomial $p(x) = c_1 + c_2x + c_3x^2$. What does the previous observation say about α, β, γ in terms of $p(x)$?

(d) (1 extra credit point) How many roots can $p(x)$ have? Explain how this shows that the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of A are linearly independent if α, β, γ are all distinct from one another.