Homework #6 - Spring 2021 MATH 185-008
Introduction to Complex Analysis

If you use an exercise that has not been shown on a previous assignment or in class, prove it first before applying it.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be given by
$$f(x) = \frac{1}{x^2+1}.$$
Why does the power series of f centered at $x = 0$ not converge for all $x \in \mathbf{R}$ despite $f \in C^\infty(\mathbf{R})$?

2. (a) Find
$$\oint_{\partial B_1(0)} \frac{\sin z}{z^{29}}\,dz.$$
(b) Find
$$\oint_{\partial B_1(0)} \left(\frac{z-2}{2z-1}\right)^3 dz.$$

3. Show
$$\oint_{|z-3|=2} \frac{e^{-z^2}}{z^3 - 9z^2 + 11z + 21}\,dz = -\frac{1}{16}\oint_{|z-3|=2} \frac{e^{-z^2}}{z-3}\,dz.$$

4. Let $z \in \mathbf{C}$ where $|z| < 2$.
Compute
$$\oint_{\partial B_2(0)} \frac{|dw|}{|z-w|^2}.$$
in terms of z by integrating over $\partial B_1(0)$.

5. Where is the function
$$f(z) = \begin{cases} \dfrac{\cos z - 1}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$
holomorphic on $\mathbf{C}$?

6. Let $0 < a < b$, $f : S \to \mathbf{C}$ be continuous on S where
$$S = \{z \mid a < \max(|\operatorname{Re}(z)|, |\operatorname{Im}(z)|) < b\}.$$
Define
$$S_r = \{z \mid \max(|\operatorname{Re}(z)|, |\operatorname{Im}(z)|) = r\}$$
and suppose
$$\oint_{S_r} f(z)\,dz = 0$$
for all $r \in (a, b)$. Prove or disprove that f is holomorphic on S.

7. Let $\Omega = B_1(0)$ and $f \in H(\Omega)$.

   Show for n-large

$$|f^{(n)}(z)| \leq n!n^n$$

   for all $z \in \Omega$. (Here n can depend on z)

8. (a) Find all holomorphic functions F on $\Omega = \{z \mid |z| < r\}$ such that for any $a \in \Omega$,

$$F(a) - F(0) = \int_0^a \overline{z}\, dz.$$

   (b) Consider the following argument:

   "Given

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

   for all z then

$$e^{\overline{z}} = \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!}.$$

   Since we have a series expansion for all z, the function

$$f(z) = e^{\overline{z}}$$

   is entire."

   Is it correct? Explain.