PHYS 341/PHYS 523/MBB 523/CBB 523/ENAS 541 Biological Physics

Problem Set 5 – Spring 2021
Due: 11:59pm on Friday March 19

Problems

Problem 1: Hydrodynamic and Diffusive Conductance through a cylindrical pipe

Consider a pipe of length L and radius R. In class we argued that it's diffusive conductance should scale like $g^{diff} \sim DR^2/L$ while its hydrodynamic conductance should scale like $g^{hydro} \sim R^4/\eta L$ with D the diffusion constant and η the diffusive conductance. Here we will calculate the prefactors more carefully, and compare the two more mathematically. For this problem consider z to parameterize the length of the pipe ($0 < z < L$) while r, θ parametrize the interior of the pipe ($r < R$).

In class we argued that both the hydrostatic pressure P and the particle density ρ obey Laplace's equation. Thus for diffusive dynamics in steady state $\nabla^2\rho = 0$ while for hydrodynamics $\nabla^2 P = 0$. Diffusive dynamics are defined by $\vec{J} = -D\nabla\rho$ while hydrodynamics are defined by $\nabla^2\vec{V} = (1/\eta)\nabla P$

(a) To calculate the diffusive conductance we consider $\rho(z = 0) = \rho_o$ while $\rho(z = L) = \rho_1$. What Boundary conditions are appropriate at the edge of the cylinder, that is when $r = R$?

(b) To calculate the hydrodynamic conductance we consider $P(z = 0) = P_o$ while $P(z = L) = P_1$. What boundary conditions are appropriate at the edge of the cylinder, that is when $r = R$?

(c) Boundary conditions on all of these surfaces uniquely specify the value of P and ρ in the interior. What satisfy these boundary conditions?

(d) For the case of diffusion, if your answer to the previous question is correct, you should be able to easily solve for the current density $\vec{J}$ and resulting conductance.

(e) For the case of hydrodynamics, you still have a complicated equation to solve! You should have something of the form $\nabla^2\vec{V} = (1/\eta)\nabla P$ where P is taken from your solution above. In cylindrical coordinates, this equation must be supplemented by boundary conditions for $\vec{V}$ at $r = 0$ and at $r = R$. What are they?

(f) You can look up (or derive!) the formula for a vector Laplacian in cylindrical coordinates. Use this to solve for the velocity field $\vec{V}(r, \theta, z)$ in the pipe.

(g) Integrate this over r, θ to find the conductance g^{hydro}. (here $I = \int V_z(r, \theta, z = 0) r dr d\theta$ over the surface.)

Problem 2: A square pipe

First, repeat (a-c) from the previous problem, but now in Cartesian coordinates, for a pipe with a square cross-section with length a so that $0 < x, y < a$.

You could finish the square pipe, just as with the last problem... but it's more interesting to think about the scaling. In the case of the cylindrical pipe, we can define $\tilde{r} = r/R$ so that $0 < R < 1$.

(d) Show that the solutions you found in the last problem can be rewritten as $V_z(r, \theta, z) = R^p f^{hydro}(\tilde{r})$, $J_z(r, \theta, z) = R^q f^{diff}(\tilde{r})$. What are the functions f and the powers p and q.

(d) Explain how these powers relate to the scaling of the total conductances with R.

(d) For a square pipe, define $\tilde{x} = X/a$ so that $0 < \tilde{x}, \tilde{y} < 1$. Show that here $V_z(x, y, z) = a^p f^{hydro}(\tilde{x}, \tilde{y})$ and $J_z(x, y, z) = a^q f^{diff}(\tilde{x}, \tilde{y})$.

(d) You don't need to solve for these functions to understand how the conductances in this problem depend on a. Write and explain the dependence of the conductances g on a.