TERM TEST 2
STA452H1

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from WED, MAR 24, 2021, 10:00am
to THURS, MAR 25, 2021, 10:00am

24 hours: open book

INSTRUCTIONS
WARNING: Failure to comply 5% off.

Show all work clearly in the space provided only.

Note also that for this test there is the stipulated requirement that you sign a contract to *academic integrity* both prior to and following completion of the test. For this please refer to next page as well as final page.

Answer all five questions. All eight pages required.
All complete questions will be valued equally at (20) each. Partial grades are shown in brackets to the right of each part but partial credit will be awarded within each such part.

Q1	Q2	Q3	Q4	Q5	total

Name_____ ***Student Number***_____

REQUIRED: AGREEMENT TO ACADEMIC INTEGRITY 1

Academic integrity is a fundamental value of learning and scholarship at the University of Toronto. Participating honestly, respectfully, responsibly, and fairly in this academic community ensures that your UofT degree is valued and respected as a true signifier of your individual academic achievement.

Prior to beginning this term test, you must attest that you will follow the Code of Behaviour on Academic Matters and will not commit academic misconduct in the completion of this test.

Affirm your agreement to this by completing the following Statement:

By signing this Statement, I, _____ (print your full name here),

_____ (student number here),

agree fully to abide by the Code of Behaviour on Academic Matters. I will not commit academic misconduct, and I am aware of the penalties that may be imposed if I commit an academic offence.

The University of Toronto's Code of Behaviour on Academic Matters outlines such behaviours that constitute academic misconduct, the processes for addressing academic offences, and the penalties that may be imposed. You are expected to be familiar with the contents of this document.

Potential offences include, but are not limited to:

- Using any unauthorized aid, including a cell phone, or searching for answers online.
- Looking at someone else's answers,
- Letting someone else look at your answers.
- Working together to answer questions.
- Sharing the exam questions or discussing answers with anyone else.
- Misrepresenting your identity or having someone else complete your test or exam.

signed _____

You may now proceed.

1. Suppose that $(U_1, \ldots, U_{2n}) \sim D_{2n}(1, \ldots, 1; 1)$ and let

$$X_n = U_1 + \cdots + U_n\,,\ Y_n = U_{n+1} + \cdots + U_{2n} \quad \text{and} \quad U_n = \frac{X_n}{X_n + Y_n}\,,\ V_n = \frac{Y_n}{X_n}$$

a) Determine EX_n, EY_n and $\sigma(X_n), \sigma(Y_n)$ as well as $\sqrt{1-\rho(X_n, Y_n)^2}$.

(10)

b) Determine EU_n, EV_n and $\sigma(U_n), \sigma(V_n)$ as well as $\sqrt{1-\rho(U_n, V_n)^2}$.

(10)

2. Suppose that $Z_n, n \in \mathbb{N}$ IID $Z \sim N(0,1)$ and let $\boldsymbol{Z}_n = (Z_1, \ldots, Z_n)'$ & $R_n = |\boldsymbol{Z}_n|$.

a) For every $n \in \mathbb{N}$, let $T_n = R_n / R_{n+2}$ and obtain both ET_n and $\sigma(T_n)$.

(7)

b) With $T_n = R_n / R_{n+2}$ as above, determine the values in $0 \leq u \leq 1$ st $\sum\limits_{n=1}^{\infty} P\big(T_n \leq u\big) \leq 1$.

(7)

c) At every $n \in \mathbb{N}$ $\Big(\prod\limits_{i=1}^{n} ER_i\Big)^2 \Big/ ER_n \;=\; a_n ER_1$.
Determine both ER_1 & ER_2 as well as the numerical sequence $a_n, n \in \mathbb{N}$.

(6)

3. Suppose that $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is uniformly distributed on the triangular region in $\mathbb{R}^3$ with vertices $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

a) Let $Y = \overline{X} = (X_1+X_2+X_3)/3$ and determine both EY and $\sigma(Y)$.

(12)

b) Determine the correlation between X_1 and X_2: $\rho(X_1, X_2)$.

(8)

4. *characteristic function basics*: given any distribution $\big(\mathbb{R}^d, \mathcal{B}^d, P\big)$ with

$$c(\boldsymbol{t}) \;=\; \int_{\mathbb{R}^d} e^{i\boldsymbol{t}'\boldsymbol{x}} dP(\boldsymbol{x}) \quad \forall\, \boldsymbol{t} \in \mathbb{R}^d$$

$$P(\boldsymbol{a}, \boldsymbol{b}] \;=\; \lim_{n\to\infty} \left(\frac{1}{2\pi}\right)^d \int_{\boldsymbol{a}}^{\boldsymbol{b}} \int_{\mathbb{R}^d} e^{-i\boldsymbol{t}'\boldsymbol{x}} c(\boldsymbol{t}) e^{-|\boldsymbol{t}|^2/2n^2} d\boldsymbol{t}\; d\boldsymbol{x} \quad \forall\, \boldsymbol{a} \le \boldsymbol{b} \;\; st \;\; P\big(\partial(\boldsymbol{a}, \boldsymbol{b}]\big) = 0.$$

verify:

a) $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y} \;\Leftrightarrow\; \boldsymbol{t}'\boldsymbol{X} \stackrel{d}{=} \boldsymbol{t}'\boldsymbol{Y} \qquad \forall\, \boldsymbol{t} \in \mathbb{R}^d$

$\Leftrightarrow\; \boldsymbol{u}'\boldsymbol{X} \stackrel{d}{=} \boldsymbol{u}'\boldsymbol{Y} \qquad \forall\, \boldsymbol{u} \in \mathbb{S}^{d-1}$

(10)

b) $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y} \;\Leftrightarrow\; c_{(\boldsymbol{X},\boldsymbol{Y})}(\boldsymbol{s}, \boldsymbol{t}) \;=\; c_{\boldsymbol{X}}(\boldsymbol{s}) c_{\boldsymbol{Y}}(\boldsymbol{t}) \quad \forall\, \boldsymbol{s} \in \mathbb{R}^r,\; \boldsymbol{t} \in \mathbb{R}^s$

(10)

5. *characteristic function basics cont'd*: given distribution $\big(\mathbb{R}^d, \mathcal{B}^d, P\big)$ with

$$c(\boldsymbol{t}) \;=\; \int_{\mathbb{R}^d} e^{i\boldsymbol{t}'\boldsymbol{x}} dP(\boldsymbol{x}) \quad \forall\, \boldsymbol{t} \in \mathbb{R}^d$$

$$P(\boldsymbol{a}, \boldsymbol{b}] \;=\; \lim_{n\to\infty} \left(\frac{1}{2\pi}\right)^d \int_{\boldsymbol{a}}^{\boldsymbol{b}} \int_{\mathbb{R}^d} e^{-i\boldsymbol{t}'\boldsymbol{x}} c(\boldsymbol{t}) e^{-|\boldsymbol{t}|^2/2n^2} d\boldsymbol{t}\; d\boldsymbol{x} \quad \forall\, \boldsymbol{a} \le \boldsymbol{b} \;\; st \;\; P\big(\partial(\boldsymbol{a}, \boldsymbol{b}]\big) = 0.$$

Verify that $c(\boldsymbol{t})$ is *integrable* wrt $d\boldsymbol{t}$ $\Leftrightarrow$ P *abs. cont.*

w. pdf $f(\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-i\boldsymbol{t}'\boldsymbol{x}} c_{\boldsymbol{X}}(\boldsymbol{t}) d\boldsymbol{t}.$

REQUIRED: AGREEMENT TO ACADEMIC INTEGRITY 2

All suspected cases of academic dishonesty will be investigated following the procedures outlined in the Code of Behaviour on Academic Matters.

In order to complete this term test, you are here required to sign the Statement below.

By signing this Statement, I , _____ (print your full name here),

_____ (student number here),

am attesting to the fact that I have abided fully to the Code of Behaviour on Academic Matters. I have not committed academic misconduct, and am aware of the penalties that may be imposed if I have committed an academic offence.

signed _____

Thank you.